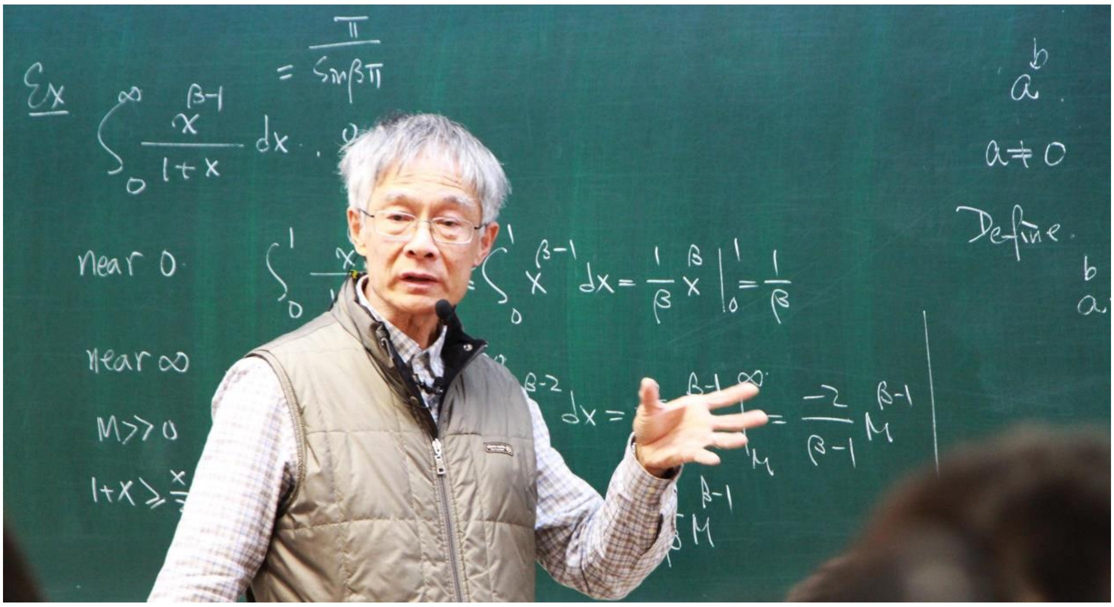
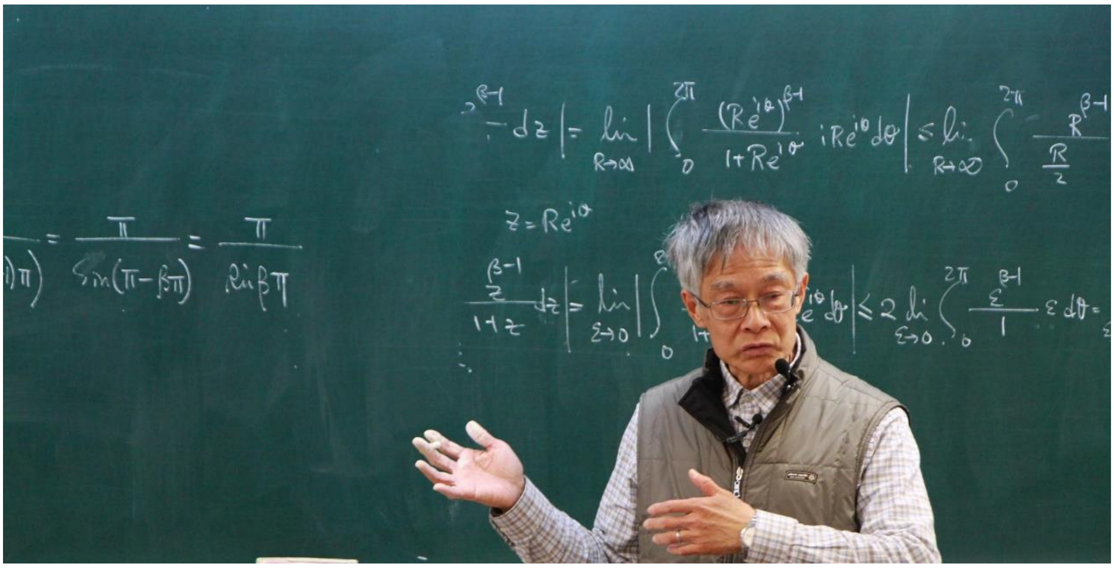
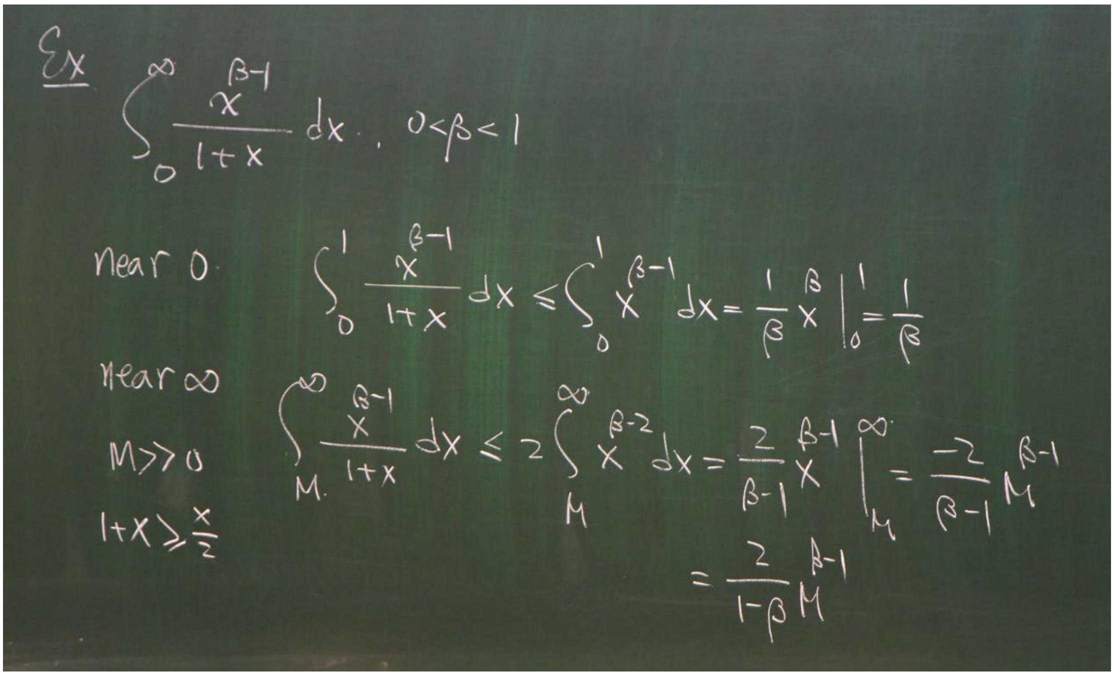
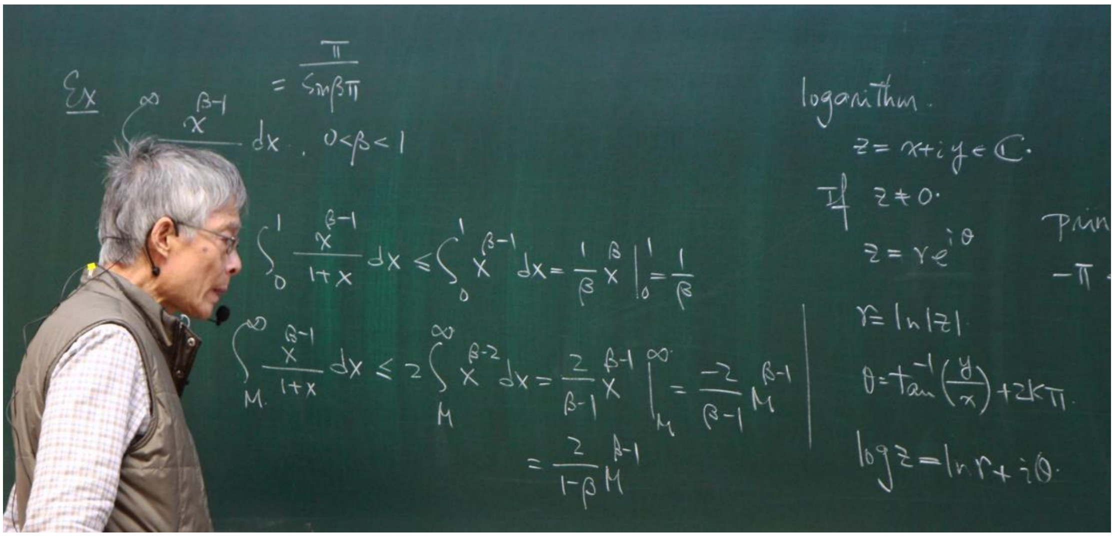
【10920 程守慶教授複變數函數論 / 第 9 堂版書】






Ex
$\int_0^\infty \frac{x^{\beta-1}}{1+x}\,dx, \quad 0<\beta<1$
near 0
$\int_0^1 \frac{x^{\beta-1}}{1+x}\,dx \le \int_0^1 x^{\beta-1}\,dx = \frac{1}{\beta}x^{\beta}\Big|_0^1 = \frac{1}{\beta}$
near ∞
M>>0
$1+x \ge \frac{x}{2}$
$\int_M^\infty \frac{x^{\beta-1}}{1+x}\,dx \le 2\int_M^\infty x^{\beta-2}\,dx = \frac{2}{\beta-1}x^{\beta-1}\Big|_M^\infty = \frac{-2}{\beta-1}M^{\beta-1}$
$= \frac{2}{1-\beta}M^{\beta-1}$


Ex
$\int_0^\infty \frac{x^{\beta-1}}{1+x}\,dx = \frac{\pi}{\sin\beta\pi}, \quad 0<\beta<1$
$\int_0^1 \frac{x^{\beta-1}}{1+x}\,dx \le \int_0^1 x^{\beta-1}\,dx = \frac{1}{\beta}x^{\beta}\Big|_0^1 = \frac{1}{\beta}$
$\int_M^\infty \frac{x^{\beta-1}}{1+x}\,dx \le 2\int_M^\infty x^{\beta-2}\,dx = \frac{2}{\beta-1}x^{\beta-1}\Big|_M^\infty = \frac{-2}{\beta-1}M^{\beta-1}$
$= \frac{2}{1-\beta}M^{\beta-1}$
logarithm.
$z = x+iy \in \mathbb{C}$.
If $z \neq 0$.
$z = re^{i\theta}$
$r = \ln|z|$
$\theta = \tan^{-1}\left(\frac{y}{x}\right) + 2k\pi$
$\log z = \ln r + i\theta$.

Ex

$$\int_0^\infty \frac{x^{\beta-1}}{1+x}\,dx, \quad 0<\beta<1 \qquad = \frac{\pi}{\sin\beta\pi}$$

near 0. $\displaystyle\int_0^1 \frac{x^{\beta-1}}{1+x}\,dx \leq \int_0^1 x^{\beta-1}\,dx = \frac{1}{\beta}x^{\beta}\Big|_0^1 = \frac{1}{\beta}$

near ∞

$M>>0$

$1+x \geq \frac{x}{2}$

$$\int_M^\infty \frac{x^{\beta-1}}{1+x}\,dx \leq 2\int_M^\infty x^{\beta-2}\,dx = \frac{2}{\beta-1}x^{\beta-1}\Big|_M^\infty = \frac{-2}{\beta-1}M^{\beta-1}$$

$$= \frac{2}{1-\beta}M^{\beta-1}$$

log

I

$\frac{1}{\beta}x^{\beta}\Big|_0^1 = \frac{1}{\beta}$

$\frac{2}{\beta-1}x^{\beta-1}\Big|_M^\infty = \frac{-2}{\beta-1}M^{\beta-1}$

$\frac{2}{1-\beta}M^{\beta-1}$

logarithm.

$z = x+iy \in \mathbb{C}$.

If $z \neq 0$.

$z = re^{i\theta}$

$r = \ln|z|$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) + 2k\pi$.

$\log z = \ln r + i\theta$.

principal value

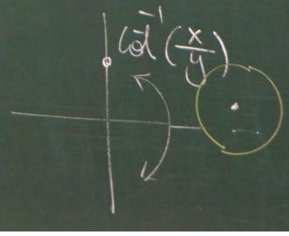
$-\pi < \theta \leq \pi$

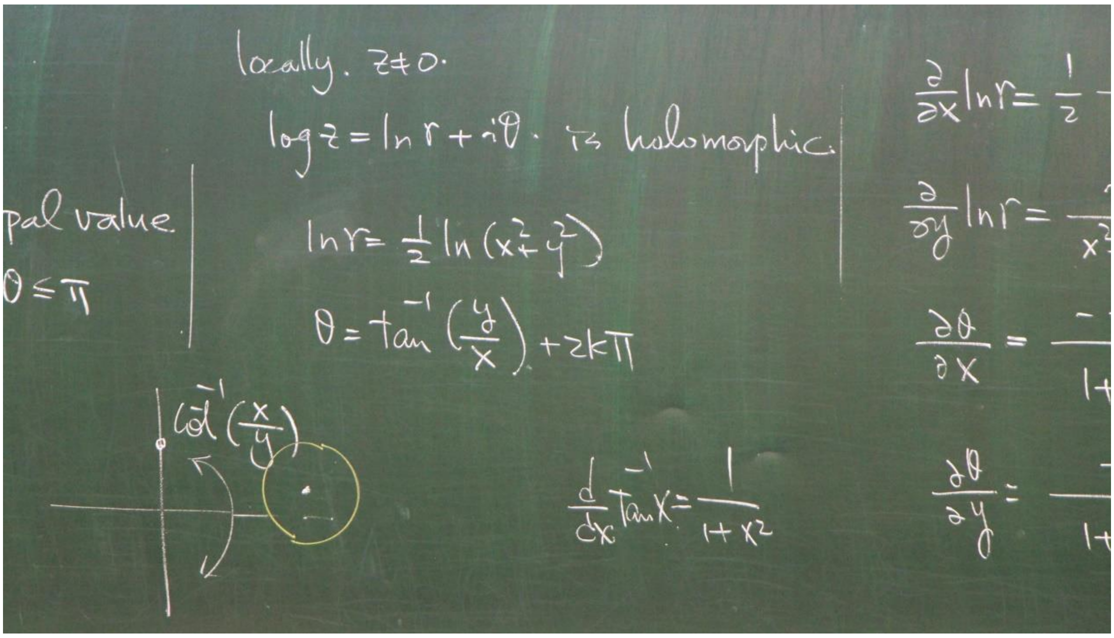
locally. $z \neq 0$.

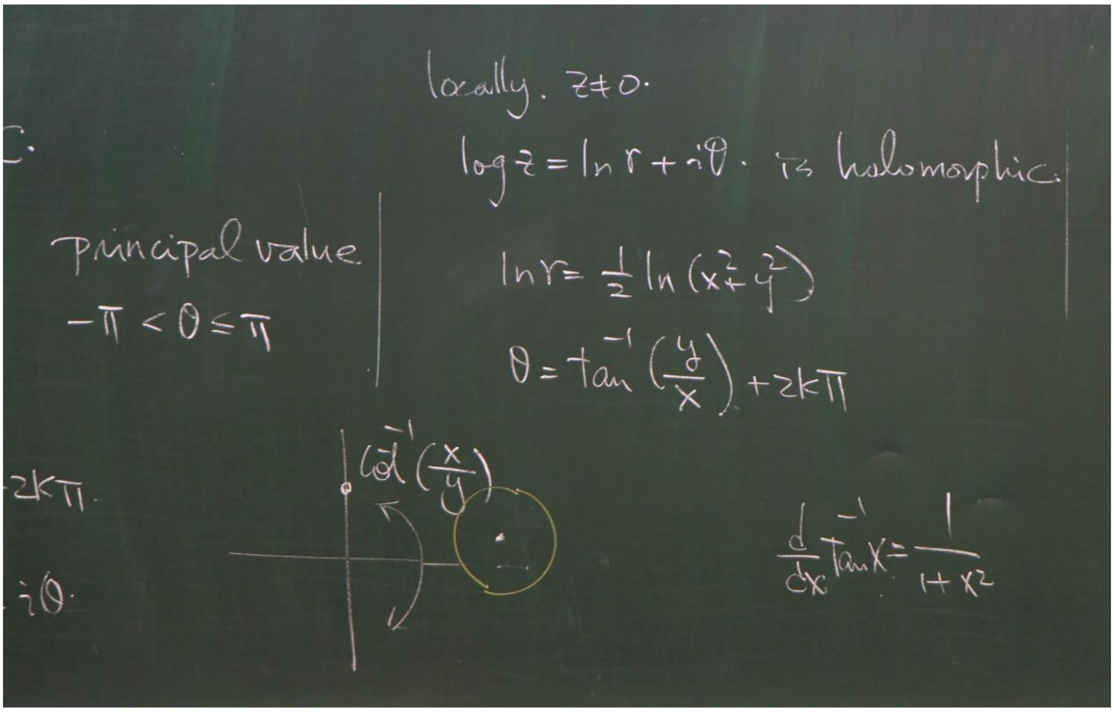
$\log z = \ln r + i\theta$

$\ln r = \frac{1}{2}\ln(x$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$




locally. $z \neq 0$.
$\log z = \ln r + i\theta$. is holomorphic.
pal value
$\theta \leq \pi$
$\ln r = \frac{1}{2}\ln(x^2+y^2)$
$\theta = \tan^{-1}\left(\frac{y}{x}\right) + 2k\pi$
$\cot^{-1}\left(\frac{x}{y}\right)$
$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$
$\frac{\partial}{\partial x}\ln r = \frac{1}{2}$
$\frac{\partial}{\partial y}\ln r =$
$\frac{\partial\theta}{\partial x} =$
$\frac{\partial\theta}{\partial y} =$


locally. $z \neq 0$.
$\log z = \ln r + i\theta$. is holomorphic.
principal value
$-\pi < \theta \leq \pi$
$\ln r = \frac{1}{2}\ln(x^2+y^2)$
$\theta = \tan^{-1}\left(\frac{y}{x}\right) + 2k\pi$
$2k\pi$.
$i\theta$.
$\cot^{-1}\left(\frac{x}{y}\right)$
$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

... holomorphic

$$<\pi$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

$$\frac{\partial}{\partial x}\ln r = \frac{1}{2}\,\frac{2x}{x^2+y^2} = \frac{x}{x^2+y^2}.$$

$$\frac{\partial}{\partial y}\ln r = \frac{y}{x^2+y^2}.$$

$$\frac{\partial\theta}{\partial x} = \frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2} = \frac{-y}{x^2+y^2}$$

$$\frac{\partial\theta}{\partial y} = \frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2} = \frac{x}{x^2+y^2}$$

$$\therefore \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

$$f'(z) = \frac{\partial}{\partial z} f(z) = u_x + i v_x$$

$$\frac{1}{2}\,\frac{2x}{x^2+y^2} = \frac{x}{x^2+y^2}.$$

$$\frac{y}{x^2+y^2}.$$

$$\frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2} = \frac{-y}{x^2+y^2}$$

$$\frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2} = \frac{x}{x^2+y^2}$$

$$\therefore \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

$$f'(z) = \frac{\partial}{\partial z} f(z) = u_x + i v_x$$

$$\frac{d}{dz}\log z = \frac{x}{x^2+y^2} + \frac{-yi}{x^2+y^2} = \frac{x-yi}{x^2+y^2} = \frac{1}{x+iy} = \frac{1}{z}$$

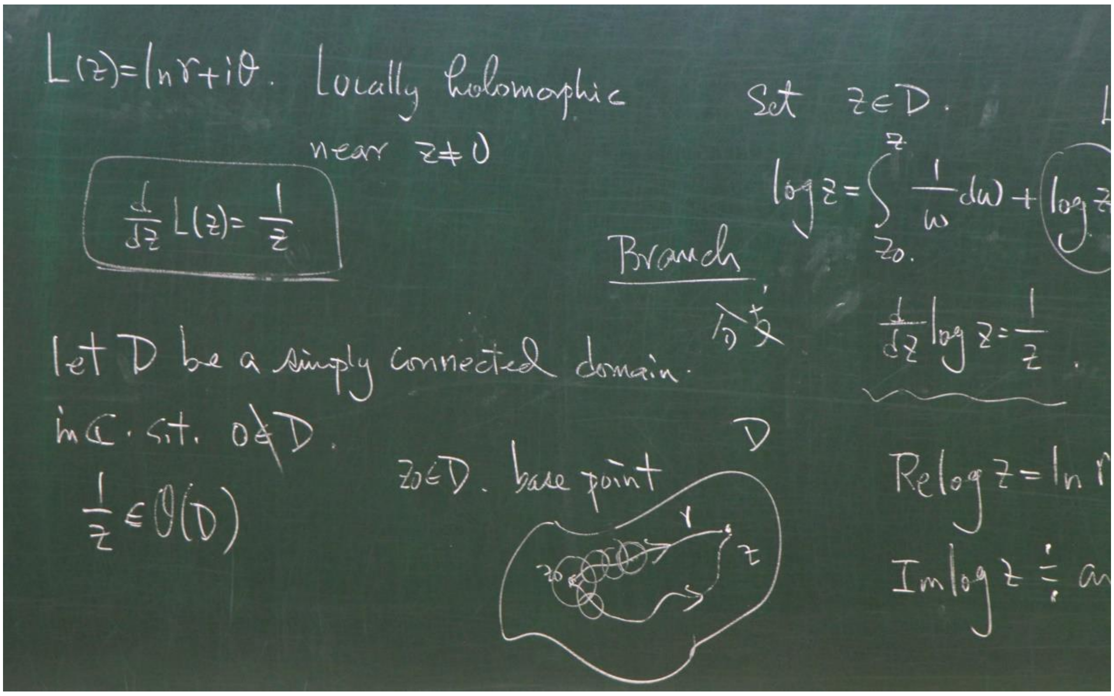
$$\frac{d}{dz}\log z = \frac{x}{x^2+y^2} + \frac{-yi}{x^2+y^2} = \frac{x-yi}{x^2+y^2} = \frac{1}{x+iy} = \frac{1}{z}$$

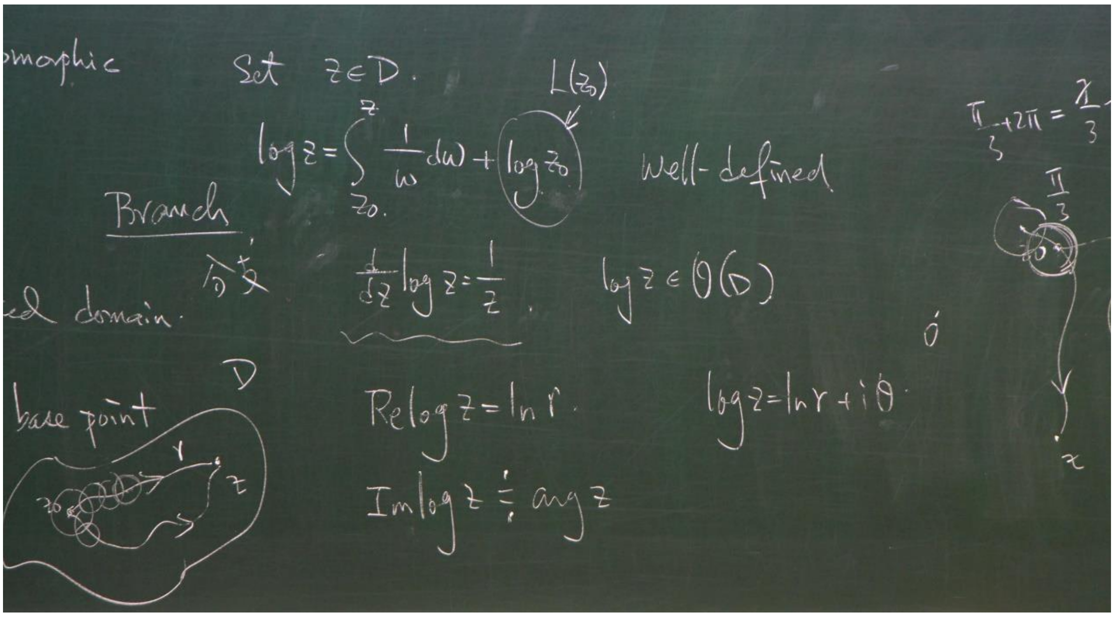
$L(z) = \ln r + i\theta$. Locally holomorphic near $z \neq 0$

$$\boxed{\frac{d}{dz}L(z) = \frac{1}{z}}$$

let D be a simply connected domain in $\mathbb{C}$ s.t. $0 \notin D$.

$$\frac{1}{z} \in \mathcal{O}(D)$$


L(z)=ln r+iθ. Locally holomorphic near z≠0
d/dz L(z)= 1/z
Branch
分支
let D be a simply connected domain in ℂ. s.t. 0∉D.
1/z ∈ O(D)
z0∈D. base point
D
γ
z0
z
Set z∈D.
log z = ∫_{z0}^{z} 1/w dw + log z0
d/dz log z = 1/z
Re log z = ln r
Im log z ≑ ar


Set z∈D.
L(z0)
log z = ∫_{z0}^{z} 1/w dw + log z0
Well-defined
Branch
分支
d/dz log z = 1/z
log z ∈ O(D)
base point
D
γ
z0
z
Re log z = ln r.
log z = ln r + iθ
Im log z ≑ arg z
π/3 + 2π =
π/3
0
x

Set $z \in D$.

$$\log z = \int_{z_0}^{z} \frac{1}{w}\,dw + \log z_0 \qquad \text{well-defined}$$

$L(z_0)$

...nch

分支

$$\frac{d}{dz}\log z = \frac{1}{z}. \qquad \log z \in \mathcal{O}(D)$$

D

$$\operatorname{Re}\log z = \ln r. \qquad \log z = \ln r + i\theta.$$

$$\operatorname{Im}\log z \doteq \arg z$$

$$\frac{\pi}{3} + 2\pi = \frac{7}{3}\pi$$

$$\frac{2x}{\ldots + y^2} = \frac{x}{x^2+y^2}.$$

$$\frac{y}{\ldots y^2}.$$

$$\frac{\ldots}{\left(\frac{y}{x}\right)^2} = \frac{-y}{x^2+y^2}$$

$$\frac{\ldots}{\left(\frac{y}{x}\right)^2} = \frac{x}{x^2+y^2}$$

D: simply-connected.

$f \in \mathcal{O}(D)$. $f(z) \neq 0$. $z \in D$.

Define.

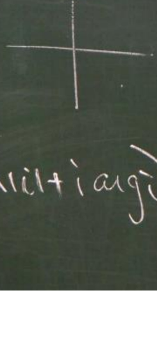
$$\log f(z) = \int_{z_0}^{z} \frac{f'(w)}{f(w)}\,dw + \log f(z_0)$$

$$\frac{d}{dz}\log f(z) = \frac{f'(z)}{f(z)}$$

a^b. $a, b \in \mathbb{C}$

$a \neq 0$ $a \neq e$

$$e^b = 1 + b + \frac{b^2}{2!} + \frac{b^3}{3!} + \cdots + \frac{b^n}{n!} + \cdots$$



Define.

$$a^b = e^{b \log a}$$

<u>Ex</u> $i^i = e^{i \log i} = e^{i(\ln|i| + i \arg i)} = e^{-\frac{\pi}{2} + 2k\pi}$ $\quad k \in \mathbb{Z}$.

$$b + \frac{b^2}{2!} + \frac{b^3}{3!} + \cdots + \frac{b^n}{n!} + \cdots$$

$\frac{\pi}{2} + 2k\pi$ $\quad k \in \mathbb{Z}$.

law of exponents

$$3^a \cdot 3^b = 3^{a+b}$$

Ex $a^b \cdot a^c$ expresses more values than a^{b+c}.

$$a^b \cdot a^c = e^{b\log a} \cdot e^{c\log a}$$

$$= e^{b(\ln|a| + i(\arg a + 2k\pi))} \cdot e^{c(\ln|a| + i(\arg a + 2m\pi))}$$

$$= |a|^{b+c} \cdot e^{i((b+c)\arg a + \underline{2kb\pi + 2mc\pi})} \qquad k, m \in \mathbb{Z}$$

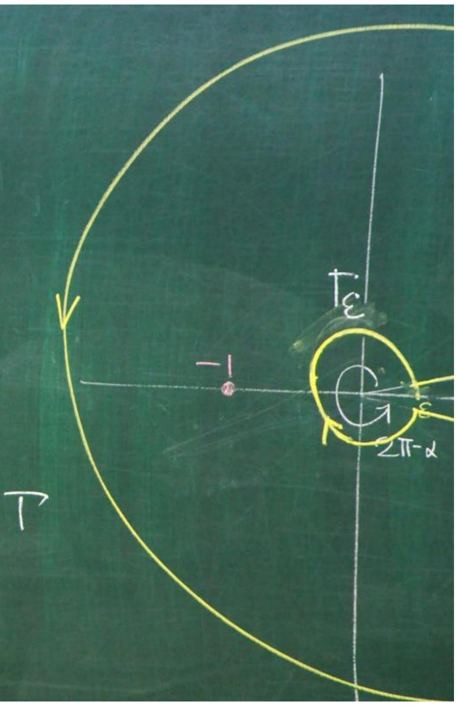
$$a^{b+c} = e^{(b+c)\log a} = e^{(b+c)(\ln|a| + i(\arg a + 2l\pi))}, \qquad l \in \mathbb{Z}$$

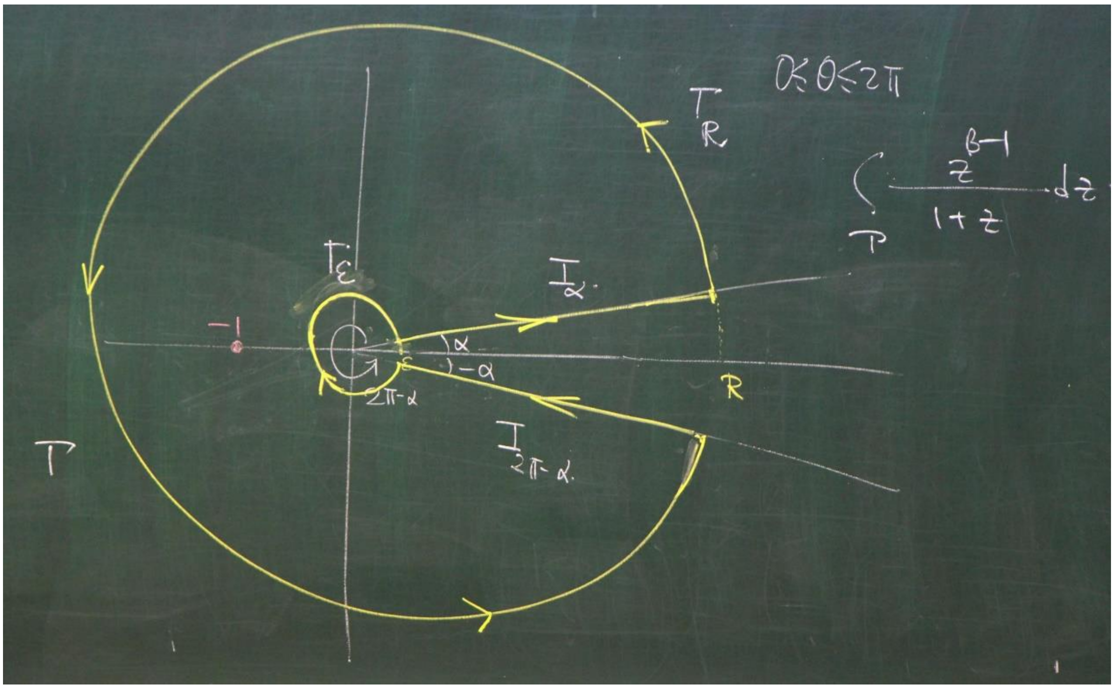
$$= |a|^{b+c} \cdot e^{i((b+c)\arg a + \underline{(b+c)2l\pi})}$$

Ex

$$\int_0^\infty \frac{x^{\beta-1}}{1+x}\,dx = \frac{\pi}{\sin\beta\pi}, \qquad 0<\beta<1$$

$$\int \frac{z^{\beta-1}}{1+z}\,dz$$





$0\leq\theta\leq 2\pi$

$$\int_{\Gamma}\frac{z^{\beta-1}}{1+z}dz=\int_{\Gamma_R}+\int_{\Gamma_\varepsilon}+\int_{I_\alpha}+\int_{I_{2\pi-\alpha}}=2\pi i\,\mathrm{Res}(f;-1)$$

$0\leq\theta\leq 2\pi$

$$\int_{\Gamma}\frac{z^{\beta-1}}{1+z}dz=\int_{\Gamma_R}+\int_{\Gamma_\varepsilon}+\int_{I_\alpha}+\int_{I_{2\pi-\alpha}}=2\pi i\,\mathrm{Res}(f;-1)$$

$$=2\pi i\, e^{i(\beta-1)\pi}$$

Fix ε, R.

let $\alpha\to 0^+$

$$\downarrow$$

$$\int_{|z|=R}+\int_{|z|=\varepsilon}$$

$$\ldots \operatorname{Res}(f;-1)$$

$$2\pi i\, e^{i(\beta-1)\pi}$$

$$\ldots\pi)) \int_\varepsilon^R \frac{x^{\beta-1}}{1+x}\,dx = 2\pi i\, e^{i(\beta-1)\pi}$$

$$\int_{I_\alpha} \frac{z^{\beta-1}}{1+z}\,dz = \int_\varepsilon^R \frac{(re^{i\alpha})^{\beta-1}}{1+re^{i\alpha}}\, e^{i\alpha}\,dr$$

$$= \int_\varepsilon^R \frac{r^{\beta-1} e^{i\alpha(\beta-1)}}{1+re^{i\alpha}}\, e^{i\alpha}\,dr$$

as $\alpha \to 0^+$

$$= \int_\varepsilon^R \frac{r^{\beta-1}}{1+r}\,dr$$

$$\int_{I_{2\pi-\alpha}} \frac{z^{\ldots}}{1+\ldots}$$

$z = re^{i\ldots}$

$r: R \to \ldots$

$dz = e^{i\ldots}$

$$\int_\Gamma \frac{z^{\beta-1}}{1+z}\,dz = \int_{\Gamma_R} + \int_{\Gamma_\varepsilon} + \int_{I_\alpha} + \int_{I_{2\pi-\alpha}} = 2\pi i \operatorname{Res}(f;-1)$$

$$= 2\pi i\, e^{i(\beta-1)\pi}$$

Fix ε, R.

let $\alpha \to 0^+$

$\downarrow$ $\alpha \to 0^+$

$$\int_{|z|=R} + \int_{|z|=\varepsilon} + \left(1 - e^{i(\beta-1)2\pi}\right) \int_\varepsilon^R \frac{x^{\beta-1}}{1+x}\,dx = 2\pi i\, e^{i(\beta-1)\pi}$$

$$\int_{I_{2\pi-\alpha}} \frac{z^{\beta-1}}{1+z}\,dz = \int_R^\varepsilon \frac{r^{\beta-1} e^{i(\beta-1)(2\pi-\alpha)}}{1+re^{i(2\pi-\alpha)}}\, e^{i(2\pi-\alpha)}\,dr$$

$$\lim_{R\to\infty} \left| \int_{|z|=\ldots} \right.$$

$z = re^{i(2\pi-\alpha)}$

$r: R \to \varepsilon$

$dz = e^{i(2\pi-\alpha)}\,dr$

as $\alpha \to 0^+$

$$= -e^{i(\beta-1)2\pi} \int_\varepsilon^R \frac{r^{\beta-1}}{1+r}\,dr.$$

$$\int_{I_{2\pi-\alpha}} \frac{z^{\beta-1}}{1+z}\,dz = \int_R^{\varepsilon} \frac{r^{\beta-1} e^{i(\beta-1)(2\pi-\alpha)}}{1+re^{i(2\pi-\alpha)}}\, e^{i(2\pi-\alpha)}\,dr \qquad \lim_{R\to\infty}\Big|\int_{|z|}$$

$z = re^{i(2\pi-\alpha)}$

$r: R \longrightarrow \varepsilon$

$dz = e^{i(2\pi-\alpha)}\,dr$

as $\alpha \to 0^+$

$$= -e^{i(\beta-1)2\pi} \int_{\varepsilon}^{R} \frac{r^{\beta-1}}{1+r}\,dr.$$

$$\lim_{R\to\infty}\Big|\int_{|z|=R} \frac{z^{\beta-1}}{z+1}\,dz\Big| = \lim_{R\to\infty}\Big|\int_0^{2\pi} \frac{(Re^{i\theta})^{\beta-1}}{1+Re^{i\theta}}\, iRe^{i\theta}\,d\theta\Big| \le \lim_{R\to\infty}\int_0^{2\pi} \frac{R^{\beta-1}}{\frac{R}{2}}\, R\,d\theta =$$

$z = Re^{i\theta}$

$$\lim_{\varepsilon\to 0}\Big|\int_{|z|=\varepsilon} \frac{z^{\beta-1}}{1+z}\,dz\Big| = \lim_{\varepsilon\to 0}\Big|\int_0^{2\pi} \frac{(\varepsilon e^{i\theta})^{\beta-1}}{1+\varepsilon e^{i\theta}}\, i\varepsilon e^{i\theta}\,d\theta\Big| \le 2\lim_{\varepsilon\to 0}\int_0^{2\pi} \frac{\varepsilon^{\beta-1}}{1}\,\varepsilon\,d\theta = \lim_{\varepsilon\to 0} 4\pi\varepsilon$$

$$\frac{z^{\beta-1}}{z+1}\,dz\Big| = \lim_{R\to\infty}\Big|\int_0^{2\pi} \frac{(Re^{i\theta})^{\beta-1}}{1+Re^{i\theta}}\, iRe^{i\theta}\,d\theta\Big| \le \lim_{R\to\infty}\int_0^{2\pi} \frac{R^{\beta-1}}{\frac{R}{2}}\, R\,d\theta = 2\lim_{R\to\infty} 2\pi R^{\beta-1} = 0$$

$|=R$

$z = Re^{i\theta}$

$$\int \frac{z^{\beta-1}}{1+z}\,dz\Big| = \lim_{\varepsilon\to 0}\Big|\int_0^{2\pi} \frac{(\varepsilon e^{i\theta})^{\beta-1}}{1+\varepsilon e^{i\theta}}\, i\varepsilon e^{i\theta}\,d\theta\Big| \le 2\lim_{\varepsilon\to 0}\int_0^{2\pi} \frac{\varepsilon^{\beta-1}}{1}\,\varepsilon\,d\theta = \lim_{\varepsilon\to 0} 4\pi\varepsilon^{\beta} = 0.$$

$|=\varepsilon.$

$$\lim_{R\to\infty}\left|\int_{|z|=R}\frac{z^{\beta-1}}{z+1}\,dz\right| = \lim_{R\to\infty}\left|\int_0^{2\pi}\frac{(Re^{i\theta})^{\beta-1}}{1+Re^{i\theta}}\, iRe^{i\theta}\,d\theta\right| \le \lim_{R\to\infty}\int_0^{2\pi}\frac{R^{\beta-1}}{\frac{R}{2}}\,R\,d\theta = 2\lim_{R\to\infty} 2\pi R^{\beta-1} = 0$$

$z = Re^{i\theta}$

$$\lim_{\varepsilon\to 0}\left|\int_{|z|=\varepsilon}\frac{z^{\beta-1}}{1+z}\,dz\right| = \lim_{\varepsilon\to 0}\left|\int_0^{2\pi}\frac{(\varepsilon e^{i\theta})^{\beta-1}}{1+\varepsilon e^{i\theta}}\, i\varepsilon e^{i\theta}\,d\theta\right| \le 2\lim_{\varepsilon\to 0}\int_0^{2\pi}\frac{\varepsilon^{\beta-1}}{1}\,\varepsilon\,d\theta = \lim_{\varepsilon\to 0} 4\pi\varepsilon^{\beta} = 0.$$

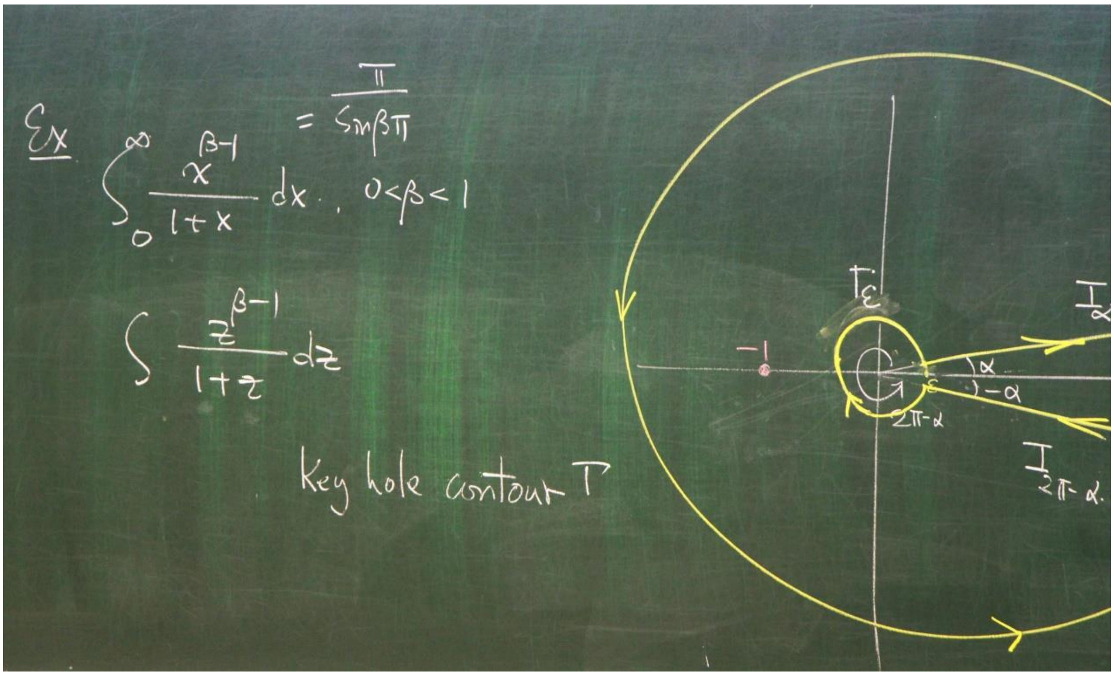
As $\varepsilon \to 0^+$. $R \to +\infty$

$$\therefore \int_0^{\infty}\frac{x^{\beta-1}}{1+x}\,dx = 2\pi i\,\frac{e^{i(\beta-1)\pi}}{1-e^{i(\beta-1)2\pi}} = 2\pi i\,\frac{1}{e^{-i(\beta-1)\pi}-e^{i(\beta-1)\pi}} = -\pi\,\frac{1}{\sin((\beta-1)\pi)} = \frac{\pi}{\sin(\pi-\beta\pi)} = \frac{\pi}{\sin\beta\pi}$$

As $\varepsilon \to 0^+$. $R \to +\infty$

$$\therefore \int_0^{\infty}\frac{x^{\beta-1}}{1+x}\,dx = 2\pi i\,\frac{e^{i(\beta-1)\pi}}{1-e^{i(\beta-1)2\pi}} = 2\pi i\,\frac{1}{e^{-i(\beta-1)\pi}-e^{i(\beta-1)\pi}} = -\pi$$

$$\frac{e^{i(\beta-1)\pi}}{1-e^{i(\beta-1)2\pi}} = 2\pi i\,\frac{1}{e^{-i(\beta-1)\pi}-e^{i(\beta-1)\pi}} = -\pi\,\frac{1}{\sin((\beta-1)\pi)} = \frac{\pi}{\sin(\pi-\beta\pi)} = \frac{\pi}{\sin\beta\pi}$$


Ex
$\int_0^\infty \frac{x^{\beta-1}}{1+x}\,dx$, $0<\beta<1$
$= \frac{\pi}{\sin\beta\pi}$
$\int \frac{z^{\beta-1}}{1+z}\,dz$
Key hole contour Γ
Γ_ε
-1
I_α
α
$-\alpha$
ε
$2\pi-\alpha$
$I_{2\pi-\alpha}$